# SELF-DUAL REPRESENTATIONS OF SL$(2, F)$ - AN APPROACH USING THE IWAHORI-HECKE ALGEBRA

KUMAR BALASUBRAMANIAN, BRAHADEESH SANKARNARAYANAN,
AND K. S. SENTHILRAANI

Abstract. Let $F$ be a non-Archimedean local field of characteristic 0 and $G = \mathrm{SL}(2, F)$. Let $(\pi, V)$ be an irreducible smooth self-dual Iwahori spherical representation $G$. The space $V$ of $\pi$ admits a non-degenerate $G$-invariant bilinear form $( \, , \, )$ which is unique up to scaling. It can be shown that the form $( \, , \, )$ is either symmetric or skew-symmetric and we set $\varepsilon(\pi) = \pm 1$ accordingly. In this paper, we use the Bernstein-Lusztig presentation of the Iwahori-Hecke algebra of $G$ and show that $\varepsilon(\pi) = 1$.

## 1. Introduction

Let $G$ be a group and $(\pi, V)$ be an irreducible complex representation of $G$. Suppose that $\pi \simeq \pi^\vee$ (here $\pi^\vee$ is the dual or contragredient representation). Using Schur's lemma, we can show that there exists a non-degenerate $G$-invariant bilinear form on $V$ which is unique up to scalars, and consequently is either symmetric or skew-symmetric. Accordingly, we set

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if the form is symmetric,} \\ -1 & \text{if the form is skew-symmetric,} \end{cases}$$

which we call the sign of $\pi$.

The sign $\varepsilon(\pi)$ is well understood for connected compact Lie groups and certain classes of finite groups of Lie type. If $G$ is a connected compact Lie group, it is known that the sign can be computed using the dominant weight attached to the representation $\pi$ (see [2] pg. 261-264). For finite groups of Lie type, computing the sign involves difficult conjugacy class computations. We refer to the following paper of Gow ([4]) where the sign is studied for such groups. In [6], Prasad introduced a nice idea to compute the sign for a certain class of representations of finite groups of Lie type. He has used this idea to determine the sign for many classical groups of Lie type. In recent times, there has been a significant interest in studying these signs in the setting of reductive p-adic groups. In [7], Prasad extended the results of [6] to the case of reductive p-adic groups and computed the sign of certain classical groups. The disadvantage of his method is that it works only for representations admitting a Whittaker model. In [8], Roche and Spallone discuss the relation between twisted sign (see section 1 in [8]) and the ordinary sign and describe a way

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of studying the ordinary sign using the twisted sign. In an earlier work ([1]), we used the ideas of Roche and Spallone ([8]) to study the sign for non-generic Iwahori spherical representations of $\mathrm{SL}(n, F)$. The key idea in this work was to reduce the problem to computing the twisted sign of a certain generic representation of a Levi subgroup of $G$ and use Prasad's method to compute the sign.

In this paper, we study this sign for Iwahori spherical representations of $\mathrm{SL}(2, F)$ using the Bernstein-Lusztig presentation of the Iwahori-Hecke algebra (explained later). To be more precise, we prove the following.

**Theorem 1.1.** *Let $G = \mathrm{SL}(2, F)$ and $(\pi, V)$ be an irreducible smooth self-dual representation of $G$ with non-trivial vectors fixed under an Iwahori subgroup. Then $\varepsilon(\pi) = 1$.*

The advantage of this method is that we don't have to restrict ourselves to any special classes of representations. In future, we hope to study the problem for $\mathrm{SL}(n, F)$ using similar techniques.

## 2. Preliminaries on signs

In this section, we briefly discuss the notion of signs associated to self-dual representations.

Let $F$ be a non-Archimedean local field and $G$ be the group of $F$-points of a connected reductive algebraic group. Let $(\pi, V)$ be a smooth irreducible representation of $G$. We write $(\pi^\vee, V^\vee)$ for the smooth dual or contragredient of $(\pi, V)$ and $\langle\, ,\, \rangle$ for the canonical non-degenerate $G$-invariant pairing on $V \times V^\vee$ (given by evaluation). Let $s : (\pi, V) \to (\pi^\vee, V^\vee)$ be an isomorphism. The map $s$ can be used to define a bilinear form on $V$ as follows

$$(w_1, w_2) = \langle w_1, s(w_2)\rangle, \quad \forall w_1, w_2 \in V.$$

It is easy to see that $(\, ,\, )$ is a non-degenerate $G$-invariant form on $V$, i.e., it satisfies,

$$(\pi(g)w_1, \pi(g)w_2) = (w_1, w_2), \quad \forall w_1, w_2 \in V.$$

Let $(\, ,\, )_*$ be a new bilinear form on $V$ defined by

$$(w_1, w_2)_* = (w_2, w_1)$$

This form is again non-degenerate and $G$-invariant. It follows from Schur's Lemma that

$$(w_1, w_2)_* = c(w_1, w_2)$$

for some non-zero scalar $c$. A simple computation shows that $c \in \{\pm 1\}$. Indeed,

$$(w_1, w_2) = (w_2, w_1)_* = c(w_2, w_1) = c(w_1, w_2)_* = c^2(w_1, w_2).$$

We set $c = \varepsilon(\pi)$. It clearly depends only on the equivalence class of $\pi$. In sum, the form $(\, ,\, )$ is symmetric or skew-symmetric and the sign $\varepsilon(\pi)$ determines its type.

## 3. THE HECKE ALGEBRA

Let $G$ be a locally profinite group and $C_c^\infty(G)$ be the space of all functions $f : G \to \mathbb{C}$ which are locally constant and compactly supported. Let $\mu$ be a Haar measure on $G$. For $f_1, f_2 \in C_c^\infty(G)$, define

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x).$$

The algebra $\mathcal{H}(G) = (C_c^\infty(G), *)$ is an associative $\mathbb{C}$ algebra and is called the Hecke algebra of $G$. For $K$ a compact open subgroup of $G$, we write $\mathcal{H}(G, K)$ for the subalgebra of $\mathcal{H}(G)$ of $K$ bi-invariant functions. To be more precise,

$$\mathcal{H}(G, K) = \{f \in \mathcal{H}(G) \mid f(k_1 g k_2) = f(g), \forall g \in G, k_1, k_2 \in K\}.$$

Let $(\pi, V)$ be a smooth representation of $G$. For $f \in \mathcal{H}(G), v \in V$, we set

$$f.v = \pi(f)(v) = \int_G f(g)\pi(g)(v) d\mu(g). \tag{3.1}$$

The above action gives $V$ the structure of a smooth $\mathcal{H}(G)$ module.

**Proposition 3.1.** *Let $(\pi, V)$ be an irreducible smooth representation of $G$ and $V^K$ be the subspace of $K$-fixed vectors in $V$. The space $V^K$ is either zero or a simple module over $\mathcal{H}(G, K)$. The process $V \to V^K$ induces a bijection between the equivalence classes of irreducible smooth representations $(\pi, V)$ of $G$ such that $V^K \neq 0$, and isomorphism classes of simple $\mathcal{H}(G, K)$-modules.*

*Proof.* We refer the reader to section 4 in [3] for a proof of the above proposition. □

## 4. THE BERNSTEIN-LUSZTIG PRESENTATION FOR THE IWAHORI-HECKE ALGEBRA

In this section we briefly explain the Bernstein-Lusztig presentation for the Iwahori-Hecke algebra of $\mathrm{SL}(2, F)$. We refer the reader to [5], for more details about the presentation in a very general setup.

Throughout, we let $G = \mathrm{SL}(2, F)$, where $F$ is a non-Archimedean local field of characteristic 0. We write $\mathfrak{o}$ for the ring of integers in $F$, $\mathfrak{p}$ for the unique maximal ideal in $\mathfrak{o}$ with generator $\varpi$ and $k_F$ for the finite residue field of cardinality $q$. Let $I$ be the subgroup of $G$ consisting of matrices of the form

$$\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{bmatrix}.$$

$I$ is called the Iwahori subgroup of $G$. We normalize the Haar measure $\mu$ such that $\mu(I) = 1$. We write $\mathcal{H} = \mathcal{H}(G, I)$ for the Iwahori-Hecke algebra of $G$. We let $T$ denote the subgroup of diagonal matrices in $G$, and let $T_\mathfrak{o} = T \cap I$. We write

$W = N_G(T)/T$ for the (finite) Weyl group, $\widetilde{W} = N_G(T)/T_{\mathfrak{o}}$ for the (infinite) affine Weyl group. Let

$$s_0 = \begin{bmatrix} 0 & \varpi \\ \varpi^{-1} & 0 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{bmatrix}.$$

It can be shown that $\widetilde{W} = \langle s_0, s_1 \mid s_0^2 = 1, s_1^2 = 1\rangle$. For $L \subset G$, we write $\chi_L$ for the characteristic function of $L$. Let $\theta = q^{-1}\chi_{IxI}$. It can be shown that $\theta$ is an invertible element in $\mathcal{H}$ and $\mathcal{A}=$ Span$\{\theta^n \mid n \in \mathbb{Z}\}$ is an abelian subalgebra of $\mathcal{H}$. For $w \in \widetilde{W}$, we let $N_w = q^{-1/2}\chi_{IwI}$.

**Proposition 4.1.** *Let* $s = s_1$ *and* $\mathfrak{B} = \{\theta^n, N_s\theta^n \mid n \in \mathbb{Z}\}$. *Then* $\mathfrak{B}$ *is a basis for* $\mathcal{H}$ *and* $\mathcal{H}$ *is generated as an algebra subject to the following relations:*

(a) $N_s^2 - \beta N_s - 1 = 0$, *where* $\beta = q^{1/2} - q^{-1/2}$.
(b) $\theta N_s - N_s\theta^{-1} = \beta(\theta + 1)$.

## 5. Reformulation using the Iwahori-Hecke algebra

In this section, we reformulate the sign of the representation in terms of the sign of a simple module over $\mathcal{H}$. To be more precise, we show that $\varepsilon(\pi)$ is the same as $\varepsilon(M)$ where $M$ is a simple module over $\mathcal{H}$.

Throughout, we let $(\pi, V)$ to be an irreducible smooth self-dual representation of $G$ with non-trivial vectors fixed under the Iwahori subgroup. We write $M = V^I$ for the subspace of vectors in $V$ fixed under $I$, $V(I) = \mathrm{Span}_{\mathbb{C}}\{\pi(k)v - v \mid v \in V, k \in I\}$. It can be shown that $V = V^I \oplus V(I)$ and $\dim_{\mathbb{C}}(M) \leq |W|$, where $W$ is the finite Weyl group generated by $s_1$. Consider the action of $\mathcal{H}$ on $V$ given in equation 3.1. Since $M \neq 0$, lemma 3.1 applies and it follows that $M$ is a simple module over $\mathcal{H}$. Let $M^\vee = \mathrm{Hom}(M, \mathbb{C})$. It can be shown that $M^\vee = (V^I)^\vee \simeq (V^\vee)^I$. For $f \in \mathcal{H}$, we set $f^\iota(g) = f(g^{-1})$. For $m^\vee \in M^\vee$ and $f \in \mathcal{H}$, define

$$(f.m^\vee)(m) = \pi^\vee(f)(m^\vee)(m) = m^\vee(f^\iota.m) = m^\vee(\pi(f^\iota)(m)). \tag{5.1}$$

It is easy to see that the above action makes $M^\vee$ a module over $\mathcal{H}$. Since $\pi \simeq \pi^\vee$, using lemma 3.1, it follows that $M \simeq M^\vee$ as simple $\mathcal{H}$ modules. Let $\tilde{T} \in \mathrm{Hom}_{\mathcal{H}}(M, M^\vee)$ be an isomorphism. As before, we define a bilinear form $((\ ,\ ))$ on $M$ as follows. For $m_1, m_2 \in M$, we set

$$((m_1, m_2)) = \langle m_1, \tilde{T}(m_2)\rangle.$$

Clearly, the above bilinear form is non-degenerate and is $\mathcal{H}$ invariant in the following sense. For $f \in \mathcal{H}, m_1, m_2 \in M$ we have

$$((f.m_1, m_2)) = ((m_1, f^\iota.m_2)).$$

For the sake of clarity, we prove the above invariance property in the following lemma.

**Lemma 5.1.** *For* $f \in \mathcal{H}$ *and* $m_1, m_2 \in M$, *we have*

$$((f.m_1, m_2)) = ((m_1, f^\iota.m_2)).$$

*Proof.* Let $\tilde{T} \in \mathrm{Hom}_{\mathcal{H}}(M, M^\vee)$. We have

$$\begin{aligned}((f.m_1, m_2)) &= \tilde{T}(m_2)(f.m_1)\\ &= (f^\iota.\tilde{T}(m_2))(m_1)\\ &= (\pi^\vee(f^\iota) \circ \tilde{T})(m_2)(m_1)\\ &= (\tilde{T} \circ \pi(f^\iota))(m_2)(m_1)\\ &= ((m_1, \pi(f^\iota)(m_2)))\\ &= ((m_1, f^\iota.m_2)).\end{aligned}$$

□

Let $((\ ,\ ))_*$ be a new bilinear form on $M$ defined by

$$((m_1, m_2))_* = ((m_2, m_1))$$

This form is again non-degenerate and $\mathcal{H}$-invariant. It follows from Schur's Lemma that

$$((m_1, m_2))_* = c((m_1, m_2))$$

for some non-zero scalar $c$. As earlier, it is easy to see that $c \in \{\pm 1\}$. We set $c = \varepsilon(M)$ and call it the sign of $M$.

It is easy to see that that $(\ ,\ )|_{M\times M}$ is non-degenerate and $\mathcal{H}$ invariant and hence it follows that $\varepsilon(\pi) = \varepsilon(M)$. We record it in the following lemma.

**Lemma 5.2.** $\varepsilon(\pi) = \varepsilon(M)$.

*Proof.* Let $w \in M$ and suppose that $(w, v) = 0, \forall v \in M$. For $x \in V(I)$, clearly we have $(w, x) = 0$. It is enough to check this when $x = \alpha - \pi(k)\alpha$, for $\alpha \in V, k \in I$. Indeed, we have

$$\begin{aligned}(w, x) &= (w, \alpha - \pi(k)\alpha)\\ &= (w, \alpha) - (\pi(k^{-1})w, \alpha)\\ &= 0.\end{aligned}$$

Now for $y = m + p \in V$, we have

$$(w, y) = (w, m) + (w, p) = 0.$$

From this it follows that $w = 0$ and $(\ ,\ )|_{M\times M}$ is non-degenerate. It is a trivial computation to check that $(\ ,\ )|_{M\times M}$ satisfies invariance property of lemma 5.1. The result follows. □

## 6. MAIN THEOREM

In this section, we prove the main result of this paper. For the sake of clarity, we recall some notation we need. We let $G = \mathrm{SL}(2, F)$ and $(\pi, V)$ an irreducible smooth self-dual Iwahori spherical representation of $G$. We write $M = V^I$ for the subspace of $V$ of vectors fixed under the Iwahori subgroup $I$ in $G$ and $(\pi, M)$ for the corresponding irreducible representation of the Iwahori-Hecke algebra $\mathcal{H}$.

Throughout we let $\mathcal{A}$ to be the abelian subalgebra of $\mathcal{H}$ as before. Since $N_s$ satisfies the quadratic relation

$$N_s^2 - \beta N_s - 1 = 0,$$

it follows that the minimal polynomial for $N_s$ (as an operator on $M$) is the polynomial $x - q^{1/2}$ or $x + q^{-1/2}$ or $(x - q^{1/2})(x + q^{-1/2})$. We consider all these cases separately. We let $M_1 = \operatorname{Ker}(N_s - q^{1/2})$ and $M_2 = \operatorname{Ker}(N_s + q^{-1/2})$.

**Lemma 6.1.** *If* $M = M_1$ *(or* $M_2$*), then* $\dim_{\mathbb{C}}(M) = 1$.

*Proof.* Consider the restriction $\pi|_{\mathcal{A}}$ of $\pi$. It is easy to see that the restriction $\pi|_{\mathcal{A}}$ is irreducible. Indeed, let $W$ be a non-zero subspace of $M$ invariant under $\mathcal{A}$ and $w \neq 0 \in W$. Since $w \in M = M_1$, we have

$$N_s.w = q^{1/2} w \in W.$$

It follows that $W$ is invariant under $\mathcal{H}$. Since $M$ is an irreducible representation of $\mathcal{H}$, we have $W = M$. Therefore $\pi|_{\mathcal{A}}$ is irreducible and $\dim_{\mathbb{C}}(M) = 1$.

□

**Lemma 6.2.** *Let* $M = M_1 \oplus M_2$*, where* $M_i \neq 0$ *for* $i = 1, 2$*. Then* $( \, , \, )|_{M_1 \times M_2} = 0$.

*Proof.* Since $-1 \in I$, it is clear that $N_s^{\iota} = N_{s^{-1}} = N_s$. Let $m_i \neq 0 \in M_i$. We have

$$q^{1/2}(m_1, m_2) = (N_s.m_1, m_2) = (m_1, N_s.m_2) = -q^{-1/2}(m_1, m_2).$$

Since $q^{1/2} + q^{-1/2} \neq 0$, it follows that $(m_1, m_2) = 0$.

□

**Lemma 6.3.** *Let* $M = M_1 \oplus M_2$*, where* $M_i \neq 0$ *for* $i = 1, 2$*. Then* $( \, , \, )|_{M_i \times M_i}$ *is non-degenerate.*

*Proof.* Suppose that $(m_1, u_1) = 0$ for all $u_1 \in M_1$. For $m \in M$, we have

$$\begin{aligned}(m_1, m) &= (m_1, u_1 + u_2)\\ &= (m_1, u_1) + (m_1, u_2)\\ &= 0.\end{aligned}$$

Since $( \, , \, )$ is a non-degenerate bilinear form on $M$, it follows that $m_1 = 0$ and the result follows.

□

**Theorem 6.4.** $\varepsilon(\pi) = 1$.

*Proof.* Since $M = V^I$, we know that $\dim_{\mathbb{C}}(M) \leq |W|$. If $M = M_1$ (or $M_2$), then lemma 6.1 applies, and it follows that the bilinear form $( \, , \, )$ on $M$ is symmetric and hence $\varepsilon(\pi) = 1$. If $M = M_1 \oplus M_2$, then we have $\dim_{\mathbb{C}}(M_i) = 1$ for $i = 1, 2$. The result now follows from lemma 6.3. □

## References

1. Kumar Balasubramanian, *Self-dual representations of* $\mathrm{SL}(n, F)$, Proc. Amer. Math. Soc. **144** (2016), no. 1, 435–444. MR 3415609
2. Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1995, Translated from the German manuscript, Corrected reprint of the 1985 translation. MR 1410059 (97i:22005)
3. C. J Bushnell and G. Henniart, *The local langlands conjecture for GL*(2), Grundlehren der Mathematischen Wissenschaften, 335 Springer - Verlag, Berlin, 2006.

4. R. Gow, *Real representations of the finite orthogonal and symplectic groups of odd characteristic*, J. Algebra **96** (1985), no. 1, 249–274. MR 808851 (87b:20015)
5. George Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), no. 3, 599–635. MR 991016
6. Dipendra Prasad, *On the self-dual representations of finite groups of Lie type*, J. Algebra **210** (1998), no. 1, 298–310. MR 1656426 (2000a:20025)
7. ______, *On the self-dual representations of a p-adic group*, Internat. Math. Res. Notices (1999), no. 8, 443–452. MR 1687319 (2000d:22019)
8. Alan Roche and Steven Spallone, *Twisted signs for p-adic linear groups*, preprint.

Kumar Balasubramanian, Department of Mathematics, IISER Bhopal, Bhopal, Madhya Pradesh 462066, India
*E-mail address*: `bkumar@iiserb.ac.in`

Brahadeesh Sankarnarayanan, Project Assistant, Department of Mathematics, IISER Bhopal, Bhopal, Madhya Pradesh 462066, India
*E-mail address*: `briha1994@gmail.com`

K. S. Senthilraani, Department of Mathematics, ISI Bangalore, Bangalore, Karnataka 560059, India
*E-mail address*: `kssenthilraani@gmail.com`